Signals & Systems formulæ

by
Satya
http://www.thesatya.com/

Contents

1 General

Energy/Power Signal

$$P = \lim_{T \to 0} \frac{1}{2\pi} \int_{-T}^{T} |g(t)^2| dt = \text{average power by } g(t) \text{ over time } T \ (2T?)$$

$$E = \int_{-\infty}^{\infty} |g(t)^2 dt = \text{total energy in g(t)}$$

Functions $f(t)$ and $g(t)$ are orthogonal in interval a to b if

$$\int_a^b f(t)g(t)dt = 0$$

2 Fourier Series

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T_0} \qquad a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_n = \frac{1}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t dt \quad b_n = \frac{1}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t dt$$

Polar form:

$$x(t) = c_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t + \phi_n)$$

$$c_0 = a_0 \qquad C_n = \sqrt{a_n^2 + b_n^2} \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

Exponential form:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad C_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 n t}dt$$

Spectrum of trignometric series exists for positive ω only.

For exponential form, for real $x(t)$, magnitude spectrum is even, phase spectrum is odd.

Magnitude spectrum -ve sign $\Rightarrow 180°$ phase shift.

dc values= a_0 S_0 C_0 etc.

Even function: $b_n = 0$, Odd function: $a_0 = a_n = 0$

S_n v/s ω :	Amplitude spectrum
ϕ_n v/s ω :	Phase spectrum

$$a_0 = c_0 \qquad a_n = (C_n + C_{-n}) \qquad b_n = j(C_n - C_{-n})$$

$$C_n = \frac{1}{2}(a_n - jb_n) \qquad C_{-n} = \frac{1}{2}(a_n + jb_n)$$

3 Fourier and Inverse Fourier Transforms

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Fourier transform exists for $f(t)$ if:

$$\text{Energy } E = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

Phase spectrum does not exist if $F(\omega)$ is real. Magnitude $M = |F(\omega)|$

$$\pi(\frac{t}{\tau}) = 1 \text{ for } -\frac{\tau}{2} < t < \frac{\tau}{2} \text{ and 0 elsewhere}$$

$$sinc(x) = \frac{\sin \pi x}{\pi x}$$

$$sgn(t) = 1 \text{ for } t > 0 \text{ and } -1 \text{ for } t < 0$$

Sampling property of unit impulse function:

$$\int_{-\infty}^{\infty} m(t)\delta(t-\tau)dt = m(\tau)$$

3.1 Properties of Fourier Transform

$$\text{Linearity: } ax(t) + by(t) \xleftrightarrow{F} aX(\omega) + bY(\omega)$$

$$\text{Time shifting: } x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(\omega)$$

$$\text{Frequency shifting: } x(t)e^{j\omega t_0} \xleftrightarrow{F} X(\omega - \omega_0)$$

$$\text{Differentiation: } \frac{d^n x(t)}{dt^n} \xleftrightarrow{F} (j\omega)^n X(\omega)$$

$$\text{Integration: } \int_{-\infty}^{t} x(t)dt \xleftrightarrow{F} \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

$$\text{Scaling: } x(at) \xleftrightarrow{F} \frac{1}{|a|} X(\frac{\omega}{a})$$

$$\text{Duality: } X(t) \xleftrightarrow{F} 2\pi x(-\omega)$$

$$\text{Convolution: } x(t) * y(t) \xleftrightarrow{F} X(\omega)Y(\omega)$$

$$F^{-1}(\frac{1}{a + j\omega}) = e^{-at}u(t)$$

$$\text{Parseval's theorem: } \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$|X(\omega)|^2 = \text{ energy density spectrum of x(t)}$$

4 z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\text{If } z = re^{j\omega} \text{ then } X(z) = z\{x(n)\} = F\{x(n)r^{-n}\}$$

If $r = 1$ then z transform reduces to Fourier transform. If ROC (Region of Convergence) of ZT includes unit circle then $x(n)$ is FTable.

$$z\{a^n u(n)\} = \frac{z}{z - a} \text{ ROC: } z > a \text{ Note special case when a=1}$$

$$z\{-a^n u(-n - 1)\} = \frac{z}{z - a} \text{ ROC: } z < a$$

4.1 Properties of z-Transform

$$\text{Linearity: } ax(n) + by(n) \overset{z}{\longleftrightarrow} aX(z) + bY(z) \text{ ROC: Intersection}$$

$$\text{Time scaling: } a_n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z) \qquad r_1 < z < r_2 \longleftrightarrow ar_1 < z < ar_2$$

$$\text{Time shifting: } x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z)$$

For ∞ duration series ROC remains same

$$\text{Time reversal: } x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \qquad r_1 < z < r_2 \leftrightarrow \frac{1}{r_1} > z > \frac{1}{r_2}$$

$$\text{Multiplication/differentiation: } nx(n) \overset{z}{\longleftrightarrow} -zX'(z)$$

$$\text{Division/integration: } \frac{x(n)}{n} \overset{z}{\longleftrightarrow} -\int_0^z \frac{X(z)}{z} dz$$

$$\text{Initial value thm: If } x(n) \text{ is causal then } x(0) = \lim_{z\to\infty} X(z)$$

$$\text{Convolution: } x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}[X^+(z) + \sum_{n=1}^{k} x(-n)z^n] \qquad k > 0$$

ROC $|z| > a_{max} \Rightarrow$ causal system

ROC $|z| < a_{min} \Rightarrow$ anticausal system

ROC includes unit circle $\Rightarrow$ stable system

5 Laplace Transform

$$L\{f(t)\} = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

$$L\{f(t)\} = F\{f(t)e^{-\sum t}\}$$

If ROC of LT includes $\sigma = 0$ then f(t) is FTable.

Derivative:

$$\frac{d}{dt}f(t) \stackrel{L}{\longleftrightarrow} aF(s) - f(0)$$

$$\frac{d^2}{dt^2}f(t) \stackrel{L}{\longleftrightarrow} a^2F(s) - sf(0) - f'(0)$$

$$\frac{d^3}{dt^3}f(t) \stackrel{L}{\longleftrightarrow} a^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\text{Integral: } \int f(t)dt \stackrel{L}{\longleftrightarrow} \frac{1}{s}F(s) + \frac{1}{s}f^{-1}(0)$$

$$\text{Time shifting: } f(t-a) \stackrel{L}{\longleftrightarrow} e^{-as}F(s)$$

$$\text{Unit impulse: } \delta(t) \stackrel{L}{\longleftrightarrow} 1$$

$$\text{Unit step: } u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{s}$$

$$t^n \stackrel{L}{\longleftrightarrow} \frac{n!}{s+n+1}$$

$$\sin\omega t \stackrel{L}{\longleftrightarrow} \frac{\omega}{s^2+\omega^2}$$

$$\cos\omega t \stackrel{L}{\longleftrightarrow} \frac{s}{s^2+\omega^2}$$

$$e^{-at} \stackrel{L}{\longleftrightarrow} \frac{1}{s+a}$$

$$\text{Initial value: } \lim_{t\to 0} f(t) \stackrel{L}{\longleftrightarrow} \lim_{s\to\infty} sF(s)$$

$$\text{Final value: } \lim_{t\to\infty} f(t) \stackrel{L}{\longleftrightarrow} \lim_{s\to 0} sF(s)$$